Lucky Imaging and Local Resolution Statistics for Atmospheric Turbulence Characterization

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Abstract: The probability of a diffraction limited image occurring in the presence of atmospheric turbulence was formulated by Fried in 1978. Implications of that analysis, regarding general statistics of the turbulence coherence diameter for a fixed aperture size, are derived as a description of "Local Resolution" statistics. Local Resolution statistics, as derived in this manner, are parameterized in the pupil plane of an optical system. However, resolution is most usefully desired in the focal plane of an optical system. We demonstrate that actual values of optical point-spread-functions (PSFs) can be used to associate, in a practical manner, Local Resolution statistics to simple focal plane measurements for resolution of images in the focal plane.

1. Introduction

On a hot summer day, a view across an expanse of landscape to distant hills reveals dynamism in the atmosphere. The heat welling-up from the ground causes every line of sight to be disturbed. The disturbances are caused by heat-driven motion that changes the index of refraction on any line of sight, making the hills in background seem to squirm or move in random, unpredictable ways, and causing the small details and features to blur. This is the visible effect of atmospheric turbulence. This is also the reason that stars seem to twinkle in the sky, for there is atmospheric turbulence present on any line of sight, no matter how strong or weak the actual source of the disturbance is. Another feature of atmospheric turbulence is that it is not the same along two lines of sight that are sufficiently separated. Thus, on the hot day, some part of the hills may be very difficult to observe while another nearby part may be sharp and well-defined. Again, this also changes constantly.

The constantly evolving nature of turbulence has led to the conclusion that conditions will occasionally occur such that the turbulence along some lines of sight could have favorable properties with respect to the quality of an image. The view of the hills in the distance could be very good and undisturbed by turbulent effects. That is, the view would be as sharp as allowed by the inherent optical parameter of the eye, camera, or telescope, i.e., the limitation of resolution for diffraction limited imaging. This phenomenon became known as a "Lucky Image".

The logical development of a Lucky Image was quantified by David Fried, in a 1978 paper that is now considered a classic in the literature of atmospheric turbulence [1]. Fried constrained the best-case resolution of a Lucky Image by the physical diffraction limit of the optics, i.e., $\alpha=\lambda/D$, where α is the resolvable angle as a function of wavelength, λ, and the diameter, D, of the optical aperture. He then set forth a simple equation that expressed the probability of a Lucky Image occurring, i.e., the probability that diffraction limited resolution is present within an image. This equation has since then been experimentally verified by measurements [2].

Fried also went one step beyond quantifying the probability of diffraction limited resolution across an image. He recognized that, even if the entire image did not achieve diffraction limited resolution, smaller regions within an image may take on many different resolution characteristics. In Fried's own words from his 1978 paper:

> "It is appropriate to note that the probability we have calculated applies independently to separate isoplanatic patches on the image. This means that in any one image, rather than its being entirely good or entirely poor resolution, there will be distributed over the image field-of-view a set of rather small regions, isoplanatic patches, in which the resolution is good." [1]

Fried identifies here "isoplanatic patches", which are regions wherein the image formation characteristics are the same. These isoplanatic patches can take on many different resolution characteristics. One patch may have good resolution, another patch may have poor resolution, a third patch may have resolution somewhere between good and poor, and so forth. Thus, Fried is describing a variation of resolutions in the image. But two natural questions arise here: 1) If resolutions vary in the image, how do we describe all these variations, and 2) What are the statistics of all these variations in resolution? Herein we present analysis to answer these questions.

2. Consequences of the Lucky Image Equation

The best possible resolution of an optical system is limited by the pupil opening that admits light (electromagnetic waves) into the optical elements that focus the light to a spot or point in the image plane. Resolution is usually expressed as the ability to resolve, i.e., separate or distinguish, two adjacent points in the optical field-of-view. The angular separation angle, α, of two points is expressed as:

$$\alpha = \frac{\lambda}{D},$$

where: λ is the mean wavelength of the light, and D is the diameter of the pupil. As D increases, the angular separation decreases and the resolution improves, as illustrated in Figure 1.

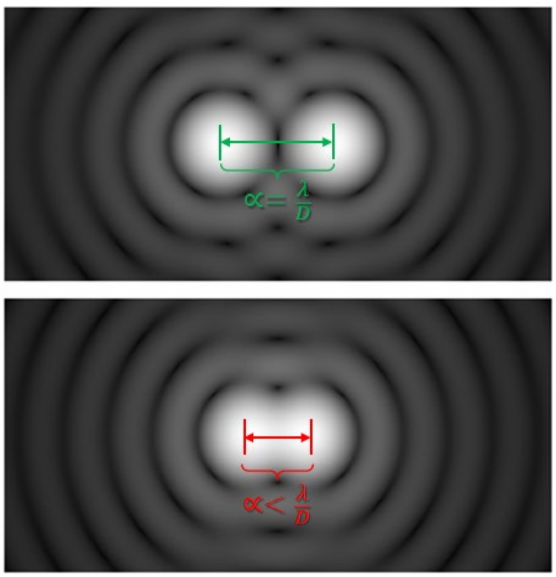

Figure 1: The ability to visually separate two point sources occurs when the angular separation reaches λ/D.

This is the diffraction limited resolution, which is the best resolution that is assumed in the Lucky Image equation analysis of Fried [1]. In the presence of atmospheric turbulence, the resolution of the aperture diameter, D, is disturbed by changes of the refraction index in the line of sight for the optical system. This has an effect equivalent to breaking the aperture D into a variety of smaller apertures. Thus, the optical system can no longer achieve the diffraction limited resolution associated with the aperture of diameter D and produces lesser resolution.

The effect of limited resolution due to turbulence was also quantified in a different paper by Fried [3]. Fried showed that the basic effects of atmospheric turbulence could be summarized by the physical quantity called the atmospheric coherence diameter, r_0. This parameter represents the size of a "patch", in the optical line of sight into the pupil, where image characteristics are the same. Therefore, as turbulence increases, the value of r_0 decreases.

Another way to think about r_0 is to consider its effect relative to angular resolution of the optical system. When $D \leq r_0$, the angular resolution is less affected by turbulence than by the small aperture's diffraction limit. However, when $D > r_0$, the image resolution is limited by a pupil aperture of diameter r_0, and the turbulence prevents the system from reaching the diffraction limit. Obviously, as r_0 decreases and turbulence worsens, the resolution decreases accordingly.

With this physical picture of how coherence diameter limits resolution, Fried then pursued the effect to a case where separate patches in the image plane behave independently with respect to resolution. This leads to Lucky Image behavior, with diffraction limited resolution in some of the separate patches and resolution worse than diffraction limited in the remaining patches. By extensive numerical analysis, Fried produced an equation for the probability that a Lucky Image will occur in any patch:

$$P_L \cong 5.6\exp\left(-0.1557(D/r_0)^2\right) \text{ , valid for } D/r_0 > 3.5 \tag{1}$$

Eq. (1) indicates the probability that diffraction limited resolution will occur as a function of the ratio of the aperture diameter to the atmospheric coherence diameter, D/r_0. As $r_0 \rightarrow D$ the probability of diffraction limited resolution approaches 1. As $r_0 \rightarrow 0$, the probability of diffraction limited resolution approaches 0. Clearly, this behavior is exactly as expected from the physical picture that was quoted from Fried above.

The distinction that we now wish to draw is the limited behavior described by Eq. (1). It describes only one probability, i.e., the probability of diffraction limited resolution. We contrast this with the description, in Fried's own words cited above, of when diffraction limited resolution is *not* achieved across the entire image, i.e., the case where:
"in any one image, rather than its being entirely good or entirely poor resolution, there will be distributed over the image field-of-view a set of rather small regions, isoplanatic patches, in which the resolution is good."

This is the consequence of the Lucky Imaging equation we focus on herein: some regions are good, some are bad, some are in-between, i.e., there are regions with many different resolution characteristics. Therefore: What are the probabilities of resolution associated with these various regions of differing resolution? Fried's analysis for the Lucky Image contains the information to describe these variations in resolution, as we demonstrate next.

2.1 Image Resolution Probabilities

The probability expressed in Eq. (1) can be given a broader interpretation than the original discussion of Fried. Consider the basic definition of a probability. Suppose we observe a random event, e.g., the toss of a gaming die. We express a specific probability associated with die taking on the value N, as P_N. However, all possible cases in the random event have a specific probability, and the sum of all the probabilities must be unity. Since the probability of the event taking the value N is given by P_N, then the probability of the die taking any value that is *not* the value N must be given by:

$$P_{\tilde{N}} = 1 - P_N \tag{2}$$

This simple law of probabilities is very useful in the context of Fried's Lucky Image equation. The probability P_L, given in Eq. (1), expresses the probability that the many independent "separate" patches, described above by Fried, take on diffraction limited properties. But the simple reasoning, summarized in Eq, (2) above, leads to the statement:

$$P_R = 1 - P_L \tag{3}$$

where Eq. (3) states the probability of a region having resolution that is *not* diffraction limited, P_R. Further, we know, from the basic facts of optical image formation, that the *best* resolution that is normally[1] achieved in the focal plane is the diffraction limited resolution. Thus, we can state for any region R, such as identified by Fried [1],

$$P_R = Prob(\text{Resolution of Region } R < \text{ Diffraction Linited}). \tag{4}$$

To summarize: Eq. (1) and Eq. (3) are complementary descriptions of the state of an image divided up into the "separate" regions of Fried. Eq. (1) gives the probability that separate regions have diffraction limited behavior, thus making the image a "Lucky Image", and Eq. (3) is the probability of any region being less than diffraction limited in resolution. Together, P_R and P_L define the probabilities of different degrees of resolution that are present everywhere in the image. Therefore, it is justified to adopt the following addition to Fried's characterizations and terminology. Since Fried called P_L the "Lucky Image" probability, we will adopt the label of "Unlucky Image" probability for P_R.

2.2 Unlucky Image Probability Distribution and Density

Eq. (1) and (3) express the probability associated with specific values of the independent variables D and r_0. However, of more interest than a single probability is the function that generates those probabilities from values of D and r_0. The exponential function, of Eq. (1), is an expression that Fried made as a fit from numerical evaluations of aberrations in the optical pupil and is valid only for values of $D/r_0 > 3.5$. The data for that fit was presented by Fried in his paper. We used the values of P_L tabulated in Fried's 1978 paper to make our own polynomial fit to *all* the data in Fried's paper. Through the use of Eq. (3), this gives a polynomial description for all of Fried's data, and not just the portion of the data to which the expression in Eq. (1) above is restricted. Thus, from our fit to Fried's tabulated data, we have the behavior of P_R, as shown in Figure 2.

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[1] The qualifier "normally" above reflects the various references to the occurrence of resolution better than the diffraction limit in turbulence, primarily due to wavefront tilt bringing information from outside the entrance pupil into the pupil. However, this "super resolution" is usually found to affect only a small fraction of any resolution measurements in the image plane, and the corresponding resolution increase is modest, so we discount that prospect herein. This also conforms with the approach of Fried on which we are developing the results presented in this paper.

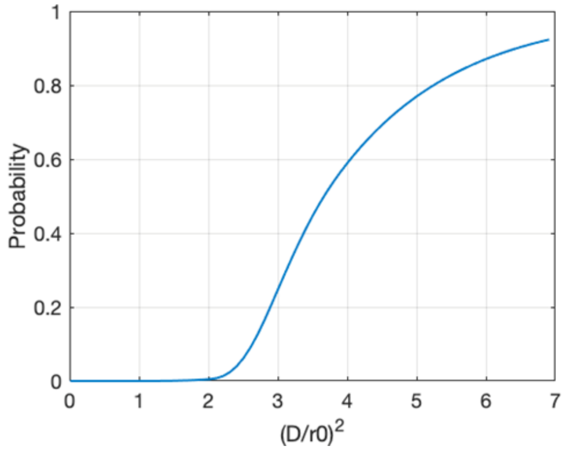

Figure 2: The probability of getting an "Unlucky Image", derived from Ref. [1].

The probability P_R, shown in Figure 2, has several implications based on the overall shape that it takes. For values of $(D/r_0)^2$ that are less than 2, the value of P_R approaches zero. To understand this limit, consider an optical system with pupil diameter, D, operated in turbulence with a coherence diameter (r_0) that approaches D. Physically, the turbulence will have less and less impact in decreasing the achievable resolution. Thus, the value of P_R should tend to zero, because P_R, as stated above in Eq. (4), is the probability that a region in the image has resolution worse than the diffraction limit. Alternatively, as the ratio D/r_0 increases, this implies the coherence diameter is decreasing, which naturally occurs as turbulence increases. The probability that a region has worse than diffraction limited resolution under these conditions increases significantly, as expected. toward an asymptote at unity.

More significant. however, is the nature of a probability function taking the shape seen in Figure 2. Assume that there is a probability density function (PDF) that is compact on the real line, i.e., tends to zero except for some compact set of values of the independent variable. The integral of such a PDF is defined as the Cumulative Distribution Function (CDF) of the PDF defined as:

$$CDF(x) = \int_{-\infty}^{x} PDF(v)dv$$

Figure 2. Thus, to get the underlying PDF from a CDF, one needs only to differentiate the CDF as defined from the integral above.

The basic property of a CDF is that it expresses the probability that a random variable is less than or equal to a given value for the random variable, v: i.e., $CDF(x) = \mathrm{Prob}(x < v)$. If the random variable is the ratio $(D/r_0)^2$, then the plot seen in Figure 2 is the CDF for the probability of the ratio. Taking the numerical derivative of that function (which we can do from the corresponding polynomial fit of Eq. (3) that we discussed above) gives the plot in Figure 3. This is the PDF for the Unlucky Image probability and illustrates the range of $(D/r_0)^2$ values where we can expect the greatest rate of increase in Unlucky Images.

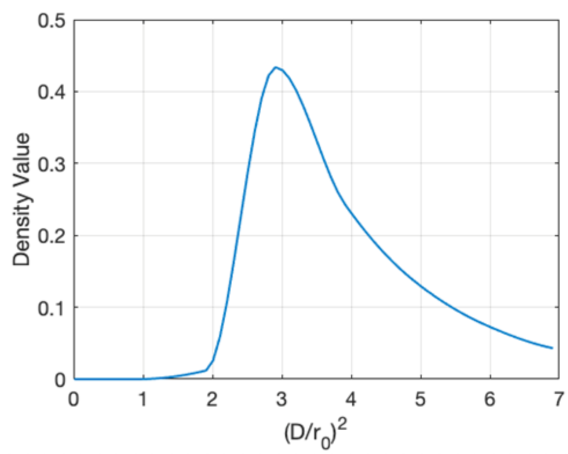

Figure 3: The probability density function (PDF) for the Unlucky Image probability suggests a particular range of D/r_o where the rate of Unlucky Images dramatically increases.

We now have a more complete characterization of the phenomena that Fried described in his 1978 paper, and we quoted above. However, we must consider an important limitation in this representation. The parameter that is random is the ratio D/r_o. The parameter r_o is a quantity that exists in the pupil plane of the optical system, and the effects of r_o are known in terms of the relationship of r_o to the pupil diameter D.

3. Image Plane Resolution Metrics

We begin by reflecting on how resolution of an image is defined and affected by the image formation behavior of an optical system. The basic measure of resolution, the Rayleigh criterion [4], is determined by how two adjacent points in the object space are visible in the image plane. The image of a point source is the point-spread-function (PSF). If two point sources are too close together, then the images of the two PSFs overlap and obscure the presence of two sources, as shown in Figure 1. Resolution of an image is a basic property tied to the PSF. The broader a PSF is, the worse is the achievable resolution in the image plane, because increasing breadth of a PSF imposes more blur of details and features are "spread" out by the image formation processes that create the PSF.

Next, we seek to construct a metric of how a PSF is "spread" out in the image plane due to turbulence. There are different ways of doing this. One metric frequently seen is the PSF encirclement. A circle of a fixed diameter is placed over the center of a PSF. Modeling of optical physics has led to the convention that the total integral of a PSF must be unity, so the image system is not modeled as losing energy in propagation from the pupil plane to the image plane. Thus, a circle of sufficiently small, and fixed, diameter will not enclose all the mass of the PSF and the total mass of the PSF will be less than unity, which we refer to as the encirclement fraction.

Clearly, encirclement fraction is a measurement of resolution. Further, it is a metric that is directly acquired in the image plane, and thus will have the desired behavior stated above: a description of resolution that does not depend on the pupil plane but exists in the image plane. For practical reasons, we will adopt a simple modification of PSF encirclement that is more common in recent years, PSF ensquarement. This is the same as PSF encirclement, except a square is placed on the center of the PSF, and the fraction of PSF mass in the square is computed. The motivation for ensquarement is simple. The prevalence of semiconductor image detectors, with rectilinear layout of the detector photosensitive wells, makes ensquarement operations into simple counting operations of the detected image pixels. [2]

The ideal approach would be to calculate ensquarement on the actual PSF of turbulence by taking the analytical or algebraic expression for a turbulent PSF produced and imposing an ensquarement upon the expression to determine the fraction of the PSF within the ensquarement. Unfortunately, this is not possible. First, turbulence consists of two general components. The first is how turbulence induces an overall tilt to the wavefront that propagates through variations the index of refraction. This tilt, arriving in the pupil plane, causes a general shift of image formation in the image plane. This is seen when the image of a PSF in turbulence wanders constantly across the image plane. The second component of turbulence relates to the higher-order aberrations that cause a random "wrinkling" of the wavefront. Analytical derivations of the average spatial frequency effects of this wavefront exist, but an analytical expression of the PSF caused by the spatial frequency disturbances has not been derived or published.

3.1 PSF Resolution in the Image Plane

Lacking an analytical calculation of turbulence ensquarement for the turbulence PSF we have adopted a data-driven approach that has proven to be useful for revealing the nature of local resolutions due to the "Unlucky Image" process in the image plane. We had access to a large set of digital image collections of PSFs taken in turbulence. The PSF collections were made by the USAF Research Laboratory at Wright-Patterson Air Force Base in September 2015. Complete details concerning the collections and various parameters can be found in a previous publication [5]. We selected a set of PSFs from this imagery that were obtained in the presence of substantial atmospheric turbulence. A scintillometer used in the experimental collections recorded values of the turbulence constant, C_n^2, in the range of 10^{-14}. This corresponds to a value for the Fried coherence diameter of 2.2 cm, which is much stronger, and more damaging to image quality, than would be acceptable in observational astronomical science.

From the selected set of PSF images we extracted a total of 800 measured PSF images. These 800 were chosen by testing individual PSFs, with a noise detection and thresholding algorithm, and retaining those PSFs that possessed sufficient signal-to-noise ratio to warrant ensquarement calculations. We then calculated the center-of-mass of each PSF and translated each so that the PSF was in the center of a frame of 65×65 pixels (at pixel coordinates 33, 33 for the center of the PSF frame). This translation also eliminated effects of wavefront tilt, leaving only the higher order aberrations that are of interest for the variations in resolution that we are considering. Squares could then be placed symmetrically around each PSF and making the ensquarement calculations simpler, accordingly.

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[2] It is also simple to relate ensquarement to encirclement, if that is preferred. A circle, inscribed within, and touching the sides of a circumscribing square, has an area that is $\pi/4$ times the area of the circumscribing square. This simple conversion factor can be used to relate any ensquarement values to encirclement values.

Figure 4 is an arbitrary selection of three of the PSFs, showing asymmetry, dispersion about the center, and random structure. These PSFs display considerable spread of optical energy from an initial point source and are not unusual as being representative of imaging conditions in situations of strong atmospheric turbulence.

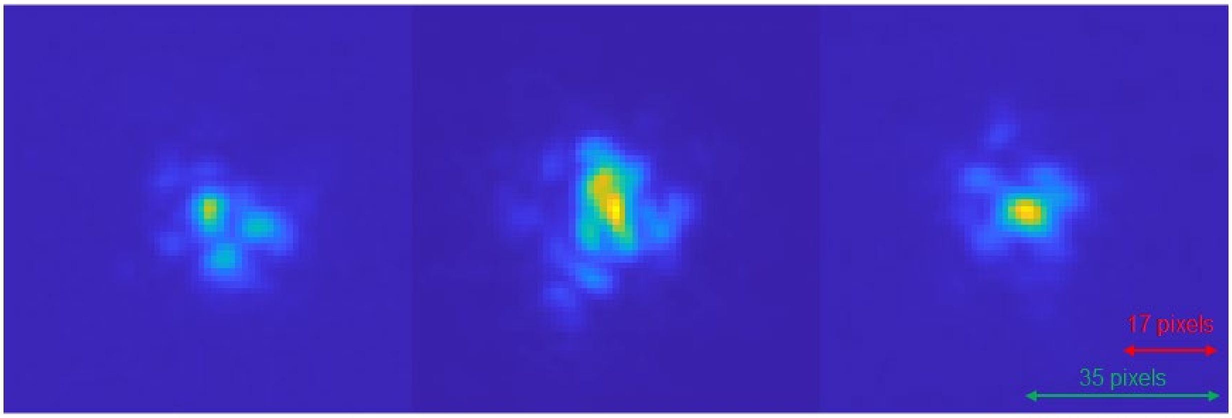

Figure 4: Three randomly selected 65x65-pixel PSF examples, illustrating the variety of turbulent effects seen in the dataset.

A square of any size can be placed around each PSF, then the fraction of its total PSF mass within that square was calculated. We call the portion of the PSF within the square the <u>*ensquarement fraction*</u>. The size of the square determines the measure of PSF resolution in the image plane. Therefore, it is important to choose a square that is large enough to encompass a substantial, but not total, portion of the PSF mass. Thus, if the square were too small, the mass in the square would tend to be small and the ensquarement fraction would show very little variation. Conversely, if the square were too large almost all the mass of the PSF would be enclosed and again ensquarement fractions would show very little variation, tending to be large and unchanging.

A "Goldilocks" middle size is desired. Some simple experiments and observations of PSFs in the total dataset led to the choice of a square of 35×35 pixels as satisfactory. An additional set of ensquarement statistics was collected for a 17×17 ensquarement, in order to provide information from a significantly smaller ensquarement as contrast to the larger ensquarement case. Size references, relative to the full 65×65 size of the three example PSFs, are shown in green and red at the right of Figure 4.

3.2 Unlucky Image Probability in the Image Plane

All 800 PSFs were ensquared in the two sizes indicated in Figure 4, and the corresponding ensquarement fractions were calculated. This ensquarement data was analyzed in two different ways. First, a histogram of the ensquarement fraction values was computed, as shown in blue in Figure 5.

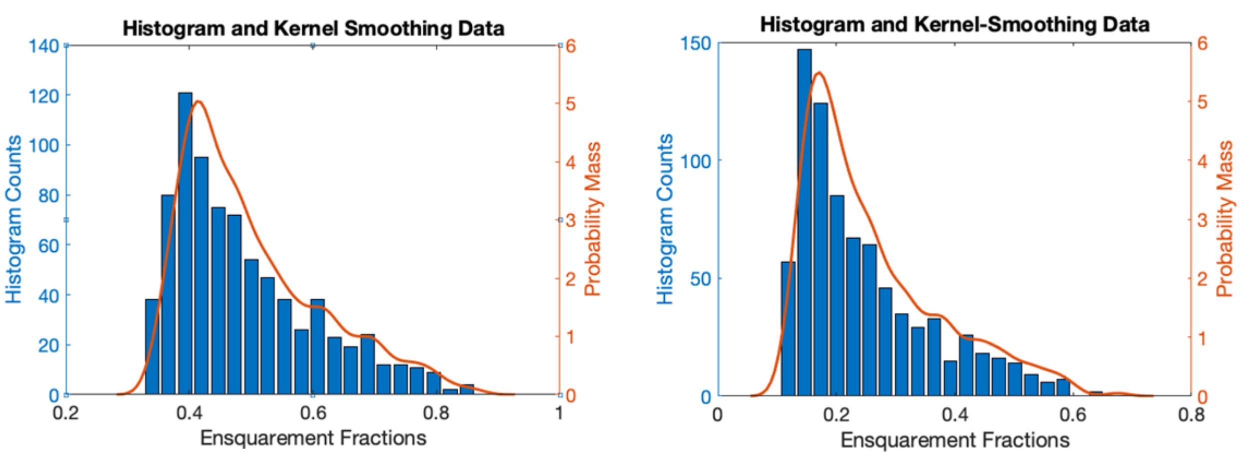

Figure 5: Ensquarement fraction histograms (blue) and empirical probability distributions (red) for 35x35-pixel squares (left) and 17x17-pixel squares (right).

A histogram can also be normalized to make a rough estimate of the Probability Density Function (PDF) that is present in a set of data, but a more sophisticated analysis is to fit a known probability distribution to the total of all the ensquarement fraction data. The shape of the histogram of the data is used as a direct guide to the type of distribution that may be used for the fit. The histograms in Figure 5 suggested that the fit be made with probability distributions having the same general shape as seen in Figure 3, i.e., a peak in the left of the distribution and a long-right-tail.

Three common distributions have such a shape: the Rayleigh distribution (fitted by one parameter) and the Gamma and Lognormal distributions (fitted by two parameters). However, the fits of these probability distributions were not satisfactory. In all three fits, to both sets of ensquarement fraction data, the left-hand peak was much broader than the peak exhibited at the left in the histograms. Further, in all three parametric fits, the right-hand-tails of the fits were not as lengthy as the tail present in the histograms computed from the data. Thus, since the fit to standard probability distributions was not successful, the decision was made to fit the ensquarement fraction data by a non-parametric probability distribution procedure. The fit was performed by the kernel-smoothing density estimation algorithm [6], shown in red in Figure 5.

4. Conclusions

The image plane results discussed in Section 3 and summarized in Figure 5 prompt the following conclusions:

- First, the ensquarement peak being in the left half of the data is due to PSFs possessing significant mass outside the ensquarement. If most of the mass of a PSF is contained in the square, then the ensquarement fraction will approach unity. Conversely, when there is substantial PSF mass inside the ensquarement, the ensquarement fraction increases. Both graphs exhibit a pronounced peak at the left, indicating PSFs in the data set are wide and dispersed across the image plane, with lesser numbers of PSFs being smaller and more contained with the ensquarement. This is not surprising, since visual inspection of PSFs show such dispersion, as exemplified in Figure 4 above.
- Second, the left-hand peaks and right-hand tails that are seen in both data sets are evidence that the squares placed on the PSFs are neither too large nor too small. If the square placed on the PSF is too small the values of ensquarement fractions will be small, nearing zero. if the PSF is too large the values of ensquarement fractions will approach one. In either case no information would be acquired about the differing resolutions that are present in a set of PSFs. Full-range statistical variability is expected for "Goldilocks" size squares. The decrease in ensquarement fraction probability in the left hand tail for the 17×17 ensquarements indicate the 17-pixel size may be too small for adequate capture of variations in image plane resolution.
- Third, the greatest difference between the plots in Figure 5 shows numerical values that are shifted to the left, for the 17×17 squares compared to the 35×35 squares. In other words, the ensquarement fractions are smaller for the 17×17 ensquarements. The fact that the two plots are similar in shape, but different in actual values, is due to the variability in PSF sizes, which directly affects the resolution of image data that would be collected with such PSFs. For a smaller square, fewer PSFs will have greater mass within the square, the opposite being true for a larger square. The shift to the left, for a smaller square is confirmation that ensquarement behaves as expected for a metric of resolution in the image plane.
- Fourth, and most significantly, actual resolutions (measured by ensquarement in the image plane) show behavior as PDFs that have the same overall shape as the PDF seen above in Figure 3. The PDF in Figure 3 was derived strictly by the mathematical consequences of probability and the published data embedded in Fried's analysis in 1978. Confirmation of Figure 3 with real PSF data is the achievement we report here.

The graphs above confirm the original insight expressed by Fried in the 1978 paper, i.e., images collected under turbulence have varying resolutions across the image plane of an optical system. Thus, an image acquired through the optical system that collected the PSF images seen above would have resolutions that change with position in the image plane. Some of those image regions would approach diffraction limited behavior, the "Lucky Images", but most would not and would be the "Unlucky Images", as we characterized them.

5. Closing Remarks

The basic "Lucky Image" formulation of Fried contained within it more than the probability of a Lucky Image. Directly from Fried's published Lucky Image data we were able to derive the probability distribution of resolutions in the many separate and independent regions of an image. Further, we are able to directly confirm these statistics of Unlucky Image resolution on the basis of the metric of ensquarement of PSFs. It should be noted that ensquarement is not the only metric of resolution that could be used for examining the distribution of Unlucky Image resolutions. A graph similar to those in Figure 5 is found in a PhD dissertation on Lucky Imaging [7] using the resolution criterion of Strehl ratio. The results we present here establish the probabilistic and mathematical connections of such empirical observations to the basic results originally presented by Fried. We observe, in closing, that Fried's analysis is in the context of constant coherence diameter, r_0, for observing of the Local Resolution statistics. However, there is always a larger ensemble of possible r_0 behavior in any "real-world" experience of turbulent atmosphere. Thus, any significant shift or change in Local Resolution statistics could be developed as a detection mechanism for time-varying turbulence strength.

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