Relative edge response and its relation to encircled energy

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Abstract

Relative edge response (RER) is a common performance metric in the remote sensing literature. RER describes the extent to which the image of a perfect edge is blurred by the optical system. It is shown that the average RER across edges in all directions is a bounded linear functional on the space of point spread functions. This formulation generalizes several common imaging performance metrics and illustrates remarkable similarity between RER and encircled energy.

1 Introduction

The National Imagery Interpretability Rating Scale (NIIRS) is a partially subjective measure of image quality widely used throughout the remote sensing community. Initially designed for military reconnaissance imagery, a trained human analyst assigns a NIIRS rating to an image based on the objects that *could* be identified in the image [1]. There exist published criteria for assigning NIIRS values. For example, to rate an image NIIRS 5, an analyst should be able to detect the presence of large individual radar antennas or an open missile silo door if such objects are present [2].

The abundance of imagery and relative scarcity of imagery analysts, along with the desire to predict NIIRS ratings prior to image collection, has led to the development of so-called image quality equations (IQEs), empirical formulae for estimating NIIRS from engineering parameters. The well-known general image quality equation (GIQE) [3] is a function of ground sample distance, signal-to-noise ratio, terms related to post-processing effects, and finally relative edge response (RER).

RER describes the extent to which the image of a perfect edge is blurred by the optical system. A perfect edge is formalized as a unit-height step function, and the RER is defined as difference in height of the blurred edge at $+\frac{1}{2}$ and $-\frac{1}{2}$ pixels, as depicted in Figure 1.

The blurring effect of an optical system is generally asymmetric, meaning that the RER will be different for edges in different directions. Averaging the RER over multiple edge directions provides a single number to use as input to an IQE.

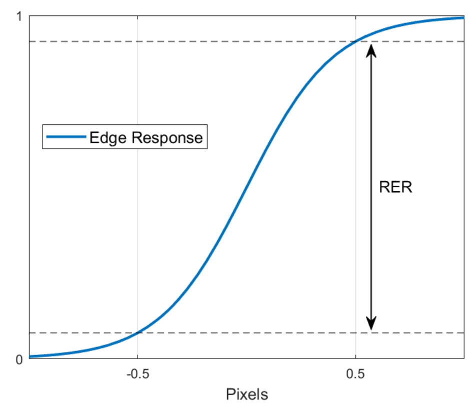

Figure 1: Relative edge response is the difference in height of a blurred edge at $\pm\frac{1}{2}$ pixels.

2 Mathematical Formulation

Let $\mathrm{psf}\colon \mathbb{R}^2 \to \mathbb{R}^+$ be the point spread function of an optical imaging system. For simplicity, $\mathrm{psf}(\mathbf{r})$ can be thought of as the irradiance at position $\mathbf{r}$ in units of pixels on the image of a distant point source. Assume $\mathrm{psf}(\mathbf{r})$ is normalized to unit integral.

An ideal edge making angle θ with the x-axis is given by

$$f(x, y) = \begin{cases} 1 & \text{if } x\cos\theta + y\sin\theta \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The edge response function $\mathcal{E}_\theta\colon \mathbb{R} \to \mathbb{R}$ is the profile of the blurred edge after f is convolved with the point spread function, and is given by

$$\mathcal{E}_\theta(d) = \int_{-d}^{\infty} dx\, \mathcal{R}_\theta\left\{\text{psf}\right\}(x), \tag{2}$$

where $\mathcal{R}_\theta\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R})$ is a projection operator, commonly known as the Radon transform:

$$\mathcal{R}_\theta\left\{\varphi\right\}(t) = \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy\, \varphi(x,y)\delta(t - x\cos\theta - y\sin\theta). \tag{3}$$

The relative edge response, as a function of edge direction θ, is defined as the difference in edge response at $+\frac{1}{2}$ and $-\frac{1}{2}$ pixels:

$$\begin{aligned} \text{rer}(\theta) &= \mathcal{E}_\theta\left(\frac{1}{2}\right) - \mathcal{E}_\theta\left(-\frac{1}{2}\right) && (4)\\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx\, \mathcal{R}_\theta\left\{\text{psf}\right\}(x). && (5) \end{aligned}$$

Equation 5 can be written as an inner product on the Hilbert space $L^2(\mathbb{R})$:

$$\text{rer}(\theta) = \left\langle \mathcal{R}_\theta\left\{\text{psf}\right\}, \text{rect}\right\rangle, \tag{6}$$

where $\text{rect}(x)$ is 1 for $|x| \leq \frac{1}{2}$ and 0 otherwise.

The central slice theorem states that the one-dimensional Fourier transform of the projection of a function at angle θ is equivalent to a slice at angle θ through the origin of the two-dimensional Fourier transform. This is expressed in operator form as $\mathcal{F}_1\mathcal{R}_\theta = \mathcal{S}_\theta\mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are the one- and two-dimensional Fourier transforms respectively, and $\mathcal{S}_\theta\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R})$ is a slice operator: $\mathcal{S}_\theta\left\{\varphi\right\}(t) = \varphi(t\cos\theta, t\sin\theta)$.

Applying the central slice theorem to Equation 6 yields an expression for RER without the projection operator $\mathcal{R}_\theta$, and with the observation that the two-dimensional Fourier transform of the point spread function is the optical transfer function, $\text{otf}\colon \mathbb{R}^2 \to \mathbb{C}$, we have

$$\begin{aligned} \text{rer}(\theta) &= \left\langle \mathcal{F}_1^{-1}\mathcal{S}_\theta\mathcal{F}_2\left\{\text{psf}\right\}, \text{rect}\right\rangle && (7)\\ &= \left\langle \mathcal{F}_1^{-1}\mathcal{S}_\theta\left\{\text{otf}\right\}, \text{rect}\right\rangle. && (8) \end{aligned}$$

The Fourier transform is a unitary operator (Parseval's theorem), meaning that $\langle f, \mathcal{F}g\rangle = \left\langle \mathcal{F}^{-1}f, g\right\rangle$, and the Fourier transform of the rect function is the well-known sinc function, therefore

$$\begin{aligned} \text{rer}(\theta) &= \left\langle \mathcal{S}_\theta\left\{\text{otf}\right\}, \text{sinc}\right\rangle && (9)\\ &= \int_{\mathbb{R}} d\xi\, \text{otf}(\xi\cos\theta, \xi\sin\theta)\,\text{sinc}(\xi). && (10) \end{aligned}$$

Thus we have an expression for the RER for an edge at a given angle θ. Equation 10 was derived in [4] using a slightly different approach. The authors explicitly derived a formula for the edge response function $\mathcal{E}$ before evaluating at $\pm\frac{1}{2}$. An advantage of the derivation presented here is that it avoids the δ distribution and Cauchy principal value associated with the Fourier transform of a step function.

3 Average RER

Let $\mathcal{M}\colon L^2(\mathbb{R}) \to \mathbb{R}$ be an *averaging* functional on the space of relative edge response functions. For example, the average RER for horizontal and vertical edges would be $\mathcal{M}\left\{\text{rer}\right\} = \frac{1}{2}\left(\text{rer}(0) + \text{rer}(\frac{\pi}{2})\right)$. In general, $\mathcal{M}$ may be nonlinear, but here we shall consider the linear operation of taking the mean value of $\text{rer}(\theta)$ on $[0, \pi)$:

$$\begin{aligned} \mathcal{M}\left\{\text{rer}\right\} &= \frac{1}{\pi}\int_0^\pi d\theta\, \text{rer}(\theta) && (11)\\ &= \frac{1}{\pi}\int_0^\pi d\theta \int_{-\infty}^{\infty} d\xi\, \text{otf}(\xi\cos\theta, \xi\sin\theta)\,\text{sinc}(\xi) && (12) \end{aligned}$$

Multiplying the integrand by $\frac{\xi}{\xi} = 1$, Equation 12 is immediately recognizable as an integral over the two-dimensional plane in polar coordinates, which again can be written as an inner product, this time on the Hilbert space $L_2(\mathbb{R}^2)$:

$$\begin{aligned} \mathcal{M}\left\{\text{rer}\right\} &= \frac{1}{\pi}\int_{\mathbb{R}^2} d^2\rho\, \text{otf}(\boldsymbol{\rho})\frac{\text{sinc}(|\boldsymbol{\rho}|)}{|\boldsymbol{\rho}|} && (13)\\ &= \left\langle \text{otf}, \frac{\text{sinc}(|\boldsymbol{\rho}|)}{\pi|\boldsymbol{\rho}|}\right\rangle && (14) \end{aligned}$$

By Parseval's theorem, Equation 14 is equal to an inner product of the PSF with some two-dimensional kernel function w:

$$\mathcal{M}\left\{\text{rer}\right\} = \langle \text{psf}, w\rangle, \tag{15}$$

where

$$w(\mathbf{r}) = \mathcal{F}_2\left\{\frac{\text{sinc}(|\boldsymbol{\rho}|)}{\pi|\boldsymbol{\rho}|}\right\}(\mathbf{r}). \tag{16}$$

Rotational symmetry allows the two-dimensional Fourier transform to be written as a Hankel transform:

$$w(r) = \mathcal{H}\left\{\frac{\text{sinc}(\rho)}{\pi\rho}\right\}(r) \tag{17}$$

$$= 2\pi \int_0^\infty d\rho\ \text{sinc}(\rho) J_0(2\pi r\rho), \tag{18}$$

where J_0 is the zeroth order Bessel function of the first kind.

The integral in Equation 18 is listed in standard reference tables [5], but it can also be solved with a simple geometrical argument. Equation 5 shows that the RER for a given edge direction is an integral of the PSF over a unit-wide infinite strip centered on the origin. Consider a test point at a distance r from the origin. The value of $w(r)$ is simply the fraction of unit-wide infinite strips containing the test point. All points inside the unit-diameter circle ($r \leq \frac{1}{2}$) will be contained in every strip. For $r > \frac{1}{2}$, the containing fraction is $\frac{\phi}{\pi}$, where ϕ is the angle formed by the two lines tangent to the unit-diameter circle and intersecting the test point, as depicted in Figure 2. Solving for ϕ in terms of r results in the following expression for $w(r)$:

$$w(\mathbf{r}) = \begin{cases} 1 & \text{if } |\mathbf{r}| \leq \frac{1}{2} \\ \frac{2}{\pi}\arcsin\left(\frac{1}{2|\mathbf{r}|}\right) & \text{if } |\mathbf{r}| > \frac{1}{2} \end{cases} \tag{19}$$

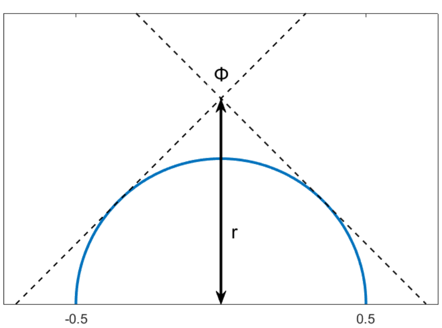

Figure 2: For $r > \frac{1}{2}$, $w(r) = \frac{\phi}{\pi}$, where ϕ is the angle depicted in the figure.

4 Relation to encircled energy

Several common imaging performance metrics can be expressed as linear functionals on the PSF. For example, the Strehl ratio S is the ratio of the peak value of the PSF to the peak value of the theoretical diffraction-limited PSF. Mathematically, this is simply a δ distribution scaled by α, the peak value of the diffraction-limited PSF:

$$S = \alpha^{-1} \int_{\mathbb{R}} d^2 r\, \text{psf}(\mathbf{r})\delta(\mathbf{r}) \tag{20}$$

Encircled energy is the integral of a properly-normalized PSF over a circle of radius a, and is often plotted as a function of a. This is expressed as an inner product $\langle \text{psf}, k\rangle$ for a kernel function k:

$$k(\mathbf{r}) = \begin{cases} 1 & \text{if } |\mathbf{r}| \leq a \\ 0 & \text{if } |\mathbf{r}| > a \end{cases} \tag{21}$$

Recall that RER is defined as the difference in edge response at $\pm\frac{1}{2}$ pixels. A generalized RER can be defined as the difference in edge response at $\pm a$ pixels, in which case w, the kernel for the inner product with the PSF, becomes

$$w(\mathbf{r}) = \begin{cases} 1 & \text{if } |\mathbf{r}| \leq a \\ \frac{2}{\pi}\arcsin\left(\frac{a}{|\mathbf{r}|}\right) & \text{if } |\mathbf{r}| > a \end{cases}. \tag{22}$$

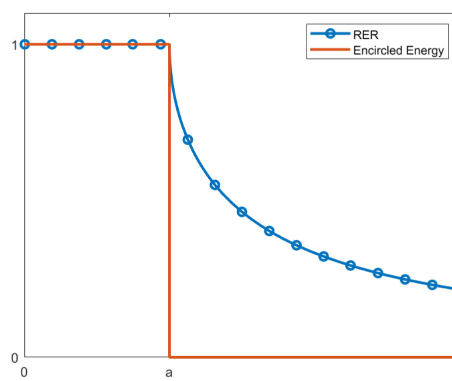

Figure 3: Radial profiles of the kernels for calculating encircled energy and generalized average RER.

Radial profiles of k and w are shown in Figure 3. Here the similarity between encircled energy and the generalized average RER is clear. As functions of a, the generalized average RER is strictly greater than the encircled energy since the PSF is nonnegative with unbounded support. However, the rapidly decreasing nature of the PSF drives the two functions to be quite similar in practice.

5 Conclusion

We have shown that the average relative edge response is a straightforward inner product of the point spread function with a rotationally symmetric kernel function. This result can have practical use in estimating RER as an input to image quality equations for predicting NIIRS ratings. Techniques for estimating RER from suitable edges within an image have been widely published [6,7]. However if no suitable edges are present, the average RER can still be calculated as long as an adequately sampled point spread function can be obtained, for example by super-resolved imaging of a distance point source or indirectly through phase retrieval techniques.

It is important to note that the average RER used here is an arithmetic mean across all edge directions. Some authors define average RER as a geometric mean of the RER in two or more edge directions. The choice of average type is often made without any real justification [8]. In such cases, engineers and IQE developers should consider defining average RER as an arithmetic mean for the tractable mathematical properties presented here.

The similarity between the average RER and encircled energy could also be of practical use to the working engineer. In cases where an encircled energy measurement is desired but difficult to obtain, a generalized average RER measurement could potentially capture much of the same information. Conversely, encircled energy could conceivably be a useful surrogate for average RER, and a requirement specification on both quantities may be largely redundant.

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